

Numerical analysis of a nonlocal parabolic problem resulting from thermistor problem^{*}

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Abstract

We analyze the spatially semidiscrete piecewise linear finite element method for a nonlocal parabolic equation resulting from thermistor problem. Our approach is based on the properties of the elliptic projection defined by the bilinear form associated with the variational formulation of the finite element method. We assume minimal regularity of the exact solution that yields optimal order error estimate. The full discrete backward Euler method and the Crank-Nicolson-Galerkin scheme are also considered. Finally, a simple algorithm for solving the fully discrete problem is proposed.

Key words: finite element method, nonlocal parabolic equation, elliptic projection, error estimates.

Mathematics Subject Classification 2000: 65M60, 65N30, 65N15.

1 Introduction

We study the numerical approximation by the finite element scheme of the nonlinear problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot (k(u) \nabla u) &= \lambda \frac{f(u)}{\left(\int_{\Omega} f(u) dx \right)^2}, \text{ in } \Omega \times]0; \bar{t}[, \\ u &= 0 \quad \text{on } \partial\Omega \times]0; \bar{t}[, \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

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where Ω is a bounded domain in \mathbb{R}^2 , \bar{t} is a positive fixed real, f and k are functions from \mathbb{R} to \mathbb{R} satisfying the hypotheses (H1) – (H2) below, λ is a positive parameter and ∇ denotes the gradient with respect to the x -variables. The time evolution model (1) describes the temperature profile of a thermistor device with electrical resistivity f , see [3,4,5,8,9,10,14,15,17,19]; the dimensionless parameter λ can be identified with the square of the applied potential difference V at the ends of the conductor. The system (1) has been the subject of a variety of investigations in the last decade. Existence of weak solutions to problems related with the thermistor problem is proved in [16], where the mathematical treatment of this system apparently appears for the first time. In [11] the problem (1) for the special case $k = 1$ is considered, and then a backward Euler time-semidiscretization method for the approximation of its solution is proposed and analyzed. In this paper we propose a finite element method to construct numerical approximations of the solutions of problem (1) for the case when k is different from the identity. The formulation of the finite element method is standard and it is based on a variational formulation of the continuous problem. There is a vast literature on finite element methods for nonlinear elliptic and parabolic problems. For example, we mention the work [7] on the porous media equations, which are similar to the Joule heating problem [1]. Compared to a standard semilinear equation, the main challenge here is the nonstandard nonlocal nonlinearity on the right-hand side of the partial differential equation (1).

On the other hand, error bounds are normally expressed in terms of norms of the exact solution of the problem. It is well known that the required regularity of the exact solution can be attained by assuming enough regularity of data, sometimes supplemented with compatibility conditions, see [2,13,20]. We then use sufficient conditions in terms of the data of the problem and its solution u that yield error estimates (see hypotheses (H1)-(H3) below).

2 Main results and organization of the paper

We denote by (\cdot, \cdot) and $\|\cdot\|$ respectively the inner product and the norm in $L^2 = L^2(\Omega)$, by $\|\cdot\|_s$ the norm in the Sobolev space $H^s(\Omega)$, by c some generic positive constant which may depend upon the data and whose value may vary from step to step. In Section 3 we study spatially semidiscrete approximations of (1) by the finite element method. The approximate solution is sought in the piecewise linear finite element space

$$S_h = S_h(\Omega) = \{\chi \in C(\Omega) : \chi|_e \text{ linear}, \forall e \in T_h; \chi|_{\partial\Omega} = 0\},$$

where $\{T_h\}_h$ is a family of regular triangulations of Ω , with h denoting the maximum diameter of the triangles of T_h . As a model for our analysis we

first consider the corresponding semidiscrete Galerkin finite element method, which consists in finding $u_h(t) \in S_h$ such that

$$\begin{aligned} (u_{h,t}, \chi) + (k(u_h) \nabla u_h, \nabla \chi) &= \frac{\lambda}{\left(\int_{\Omega} f(u_h) dx \right)^2} (f(u_h), \chi), \\ u_h(0) &= u_{0h}, \end{aligned} \quad (2)$$

$\forall \chi \in S_h$, $t \in J = (0, \bar{t})$, and where $u_{0h} \in S_h$ is a given approximation of u_0 . Similar discretization techniques have been analyzed for various linear and nonlinear evolution problems (cf. e.g. [13]). This method (2) may be written as a system of ordinary differential equations. In fact, let $\{\phi\}_{j=1}^{N_h}$ be the standard pyramid basis of S_h . Write $u_h(x, t) = \sum_{j=1}^{N_h} \alpha_j(t) \phi_j(x)$, where $(\alpha_j)_{1 \leq j \leq N_h}$ are the real coefficients to be determined. Then, (2) can be written as

$$A\alpha'(t) + B(\alpha)\alpha(t) = \tilde{f}(\alpha), \quad t \in J, \quad \alpha(0) = \gamma,$$

where γ is the vector of nodal values of u_{0h} , $\tilde{f}(\alpha) = (\tilde{f}_1(\alpha), \dots, \tilde{f}_{N_h}(\alpha))^T$ with

$$\tilde{f}_j(\alpha) = \frac{\lambda}{\left(\int_{\Omega} f(\sum_{l=1}^{N_h} \alpha_l(t) \phi_l) dx \right)^2} \left(f \left(\sum_{l=1}^{N_h} \alpha_l(t) \phi_l \right), \phi_j \right),$$

and $A = (a_{jk})_{1 \leq j, k \leq N_h}$ and $B(\alpha) = (b_{jk}(\alpha))_{1 \leq j, k \leq N_h}$ are, respectively, the corresponding mass and stiffness matrices:

$$a_{jk} = (\phi_j, \phi_k), \quad b_{jk}(\alpha) = \left(k \left(\sum_{l=1}^{N_h} \alpha_l(t) \phi_l \right) \nabla \phi_j, \nabla \phi_k \right).$$

We shall assume the following general assumptions on the given data:

- (H1)** $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitzian function and $f(u) \geq \sigma > 0$ for all $u \in \mathbb{R}$;
- (H2)** k is a twice derivable function verifying: there exist positive constants k_1, k_2 and c such that $0 < k_1 \leq k(u) \leq k_2$, $|k'(u)|, |k''(u)| \leq c$;
- (H3)** $u \in L^\infty(0, T, H^2(\Omega) \cap W^{1,\infty}(\Omega))$ and $u_0 \in H^2(\Omega)$.

It is shown in [6, Theorem 2.1] that the regularity assumption (H3) is satisfied if the data is smooth and compatible. The matrix A is always definite positive. Further, hypotheses (H1) and (H2) assure that the matrix $B(\alpha)$ is also positive definite. Assumption (H3) is used in order to prove Lemma 1 and then to show a $o(h^2)$ error estimate. It is useful to introduce the interpolation operator $I_h : C(\Omega) \rightarrow S_h$ defined by

$$I_h v = \sum_{j=1}^{N_h} v(P_j) \phi_j(x),$$

where $\{P_j\}_{j=1}^{N_h}$ are the interior vertices of T_h . For completeness, we assume the following standard interpolation error estimates : for $v \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists a positive constant $c > 0$ such that

$$\|I_h v - v\| \leq ch^2 \|v\|_2 \text{ and } \|\nabla(I_h v - v)\| \leq ch \|v\|_2,$$

holds. We also assume the property [18]: for some integer $r \geq 2$ and small h ,

$$\inf_{\chi \in S_h} \{\|v - \chi\| + h\|\nabla(v - \chi)\|\} \leq ch^s \|v\|_s, \text{ for } 1 \leq s \leq r, \quad (3)$$

when $v \in H^s \cap H_0^1$. We finally suppose that the family of triangulations is such that the inverse estimate [18]

$$\|\nabla \chi\| \leq ch^{-1} \|\chi\| \quad \forall \chi \in S_h \quad (4)$$

is satisfied. In the existing literature the error estimates for the finite element method are normally expressed in terms of norms of the exact solution of the problem and are usually derived for solutions that are sufficiently smooth (cf. e.g. [13]). To estimate the error in the semidiscrete problem (2) we split the error as $u_h - u = (u_h - \tilde{u}_h) + (\tilde{u}_h - u) = \theta + \rho$, where \tilde{u}_h denotes the standard elliptic projection in S_h of the exact solution u defined by

$$(k(u(t))\nabla(\tilde{u}_h - u), \nabla \chi) = 0, \quad \forall \chi \in S_h, \quad t \geq 0. \quad (5)$$

It is well known (see [12,18]) that if the regularity hypothesis (H3) for u holds, then \tilde{u}_h has the following approximation properties:

Lemma 1 *Under the regularity hypotheses (H1)-(H3), one has*

$$\|\rho(t)\| + h\|\nabla \rho(t)\| \leq c(u)h^2,$$

$$\|\rho_t(t)\| + h\|\nabla \rho_t(t)\| \leq c(u)h^2,$$

where $c(u)$ is a constant independent of $t \in J$.

Lemma 2 *Let \tilde{u}_h be defined by (5). Then, $\|\nabla \tilde{u}_h(t)\|_{L^\infty} \leq c(u)$, $t \in J$.*

For the finite element method (2), if u_{0h} is chosen such that $\|u_{0h} - u_0\| \leq ch^2 \|u_0\|_2$, we prove in Section 3 an error estimate of the form $\|u_h(t) - u(t)\| \leq c(u)h^2$ (Theorem 3). The corresponding estimate for the gradient is also proved (Theorem 4). In Section 4 we show that our approach for the semidiscrete Galerkin finite element method also applies to fully discrete schemes. We consider the backward Euler method for the discretization in time of (1): letting τ to be the time step, U^n the approximation in S_h of $u(t)$ at $t = t_n = n\tau$, i.e. $U^n = \sum_{j=1}^{N_h} \alpha_j^n \phi_j$, where $(\alpha_j^n)_{1 \leq j \leq N_h}$ are the unknown real coefficients,

$\partial_n U^n = \frac{U^n - U^{n-1}}{\tau}$, $n = 0, 1, 2, \dots$, the numerical method is defined by

$$\begin{aligned} (\partial_n U^n, \chi) + (k(U^n) \nabla U^n, \nabla \chi) &= \frac{\lambda}{\left(\int_{\Omega} f(U^n) dx \right)^2} (f(U^n), \chi), \quad \forall \chi \in S_h, \\ U^0 &= u_{0h}. \end{aligned} \quad (6)$$

For this scheme we prove (Theorem 5), under the same regularity requirements as in Section 3, that

$$\|U^n - u(t_n)\| \leq c(u)(h^2 + \tau), \text{ for } t_n \in J = (0, \bar{t}).$$

In order to obtain higher accuracy in time, in Section 5 we investigate an alternative way to obtain an $o(h^2 + \tau^2)$ error bound using the basic Crank-Nicolson-Galerkin scheme: applying the techniques of sections 3 and 4 we prove (Theorem 7) an error estimate of the form

$$\|U^n - u(t_n)\| \leq c(u)(h^2 + \tau^2), \text{ for } t_n \in J = (0, \bar{t}).$$

Finally, in Section 6 we propose a simple algorithm for solving the fully discrete problem.

3 Semidiscrete problem

In this section we obtain an error estimate for u and the associated estimate for the gradient. The proofs use a splitting of the error based on the elliptic projection \tilde{u}_h (5). We may define the semidiscrete problem on a finite interval $J = (0, \bar{t}]$ of time.

Theorem 3 *Let u and u_h be the solutions of (1) and (2), respectively. Then, under the hypotheses (H1)-(H3), we have: $\|u_h(t) - u(t)\| \leq c(u)h^2$, for $t \in J$, provided that $\|u_{0h} - u_0\| \leq ch^2$.*

PROOF. Owing to Lemma 1 and to the decomposition of the error as sum of two terms $u_h - u = (u_h - \tilde{u}_h) + (\tilde{u}_h - u) = \theta + \rho$, it suffices to treat $\theta = u_h - \tilde{u}_h$.

We have from the equations satisfied by u_h and \tilde{u}_h that

$$\begin{aligned}
& (\theta_t, \chi) + (k(u_h) \nabla \theta, \nabla \chi) \\
&= (u_{h,t}, \chi) + (k(u_h) \nabla u_h, \nabla \chi) - (\tilde{u}_{h,t}, \chi) - (k(u_h) \nabla \tilde{u}_h, \nabla \chi) \\
&= \frac{\lambda}{(\int f(u_h))^2} (f(u_h), \chi) - (\rho_t, \chi) - (u_t, \chi) - (k(u) \nabla \tilde{u}_h, \nabla \chi) + ((k(u) - k(u_h)) \nabla \tilde{u}_h, \nabla \chi) \\
&= \frac{\lambda}{(\int f(u_h))^2} (f(u_h), \chi) - (\rho_t, \chi) - (u_t, \chi) - (k(u) \nabla u, \nabla \chi) + ((k(u) - k(u_h)) \nabla \tilde{u}_h, \nabla \chi) \\
&= \frac{\lambda}{(\int f(u_h))^2} (f(u_h), \chi) - \frac{\lambda}{(\int f(u))^2} (f(u), \chi) + ((k(u) - k(u_h)) \nabla \tilde{u}_h, \nabla \chi) - (\rho_t, \chi) \\
&= \frac{\lambda}{(\int f(u_h))^2} (f(u_h) - f(u), \chi) + \left(\frac{\lambda}{(\int f(u_h))^2} - \frac{\lambda}{(\int f(u))^2} \right) (f(u), \chi) \\
&+ ((k(u) - k(u_h)) \nabla \tilde{u}_h, \nabla \chi) - (\rho_t, \chi) \\
&= \frac{\lambda}{(\int f(u_h))^2} (f(u_h) - f(u), \chi) + ((k(u) - k(u_h)) \nabla \tilde{u}_h, \nabla \chi) - (\rho_t, \chi) \\
&+ \frac{\lambda}{(\int f(u_h))^2 (\int f(u))^2} \left(\int_{\Omega} (f(u_h) - f(u)) dx \right) \left(\int_{\Omega} (f(u_h) + f(u)) dx \right) (f(u), \chi).
\end{aligned}$$

Thus, setting $\chi = \theta$, using the hypotheses (H1)-(H3), Lemma 1, and Young's inequality,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + k_1 \|\nabla \theta\|^2 &\leq c(\|u_h - u\|(\|\theta\| + \|\nabla \theta\|) + \|\rho_t\| \|\theta\|) \\
&\leq k_1 \|\nabla \theta\|^2 + c(\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2).
\end{aligned}$$

By integration, we get $\|\theta(t)\|^2 \leq \|\theta(0)\|^2 + c \int_0^t (\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2) ds$ and it follows by Gronwall's Lemma that

$$\|\theta(t)\|^2 \leq c\|\theta(0)\|^2 + c \int_0^t (\|\rho\|^2 + \|\rho_t\|^2) ds$$

or

$$\|\theta(0)\| \leq \|u_{0h} - u_0\| + \|\tilde{u}_h(0) - u_0\| \leq \|u_{0h} - u_0\| + ch^2 \|u_0\|_2. \quad (7)$$

We then get the desired conclusion:

$$\|\theta(t)\| \leq c\|u_{0h} - u_0\| + c(u)h^2 \leq c(u)h^2.$$

We now derive for the standard Galerkin method, from the weak formulation of the parabolic problem and using the inverse property, the optimal order error estimate for the gradient.

Theorem 4 *Let u and u_h be, respectively, the solutions of (1) and (2). Under hypotheses (H1)-(H3), if u_{0h} is chosen such that $\|u_{0h} - u_0\| \leq ch^2 \|u_0\|_2$, then*

$$\|\nabla u_h(t) - \nabla u(t)\| \leq c(u)h, \text{ for } t \in J.$$

PROOF. We have

$$\begin{aligned}
\|\nabla u_h(t) - \nabla u(t)\| &\leq \|\nabla(u_h(t) - \chi)\| + \|\nabla\chi - \nabla u(t)\| \\
&\leq ch^{-1}\|u_h(t) - \chi\| + \|\nabla\chi - \nabla u(t)\| \\
&\leq ch^{-1}\|u_h(t) - u(t)\| + ch^{-1}(\|\chi - u(t)\| + h\|\nabla\chi - \nabla u(t)\|).
\end{aligned} \tag{8}$$

By the approximation assumption (3) we know that, with suitable $\chi \in S_h$,

$$\|\chi - u(t)\| + h\|\nabla\chi - \nabla u(t)\| \leq ch^2\|u(t)\|_2.$$

Then, using (8), we get:

$$\|\nabla u_h(t) - \nabla u(t)\| \leq ch^{-1}\|u_h(t) - u(t)\| + ch\|u(t)\|_2.$$

Theorem 3 yields the intended conclusion: $\|\nabla u_h(t) - \nabla u(t)\| \leq c(u)h$.

4 The completely discrete case

We now turn our attention to the fully discrete scheme based on the backward Euler method: find $U^n \in S_h$ such that (6) holds. Existence result of (6) is a simple consequence of Brouwer's fixed point theorem. Here we obtain an error estimate for the scheme. For h , τ and $\frac{\tau}{h}$ small enough, we prove uniqueness.

Theorem 5 *Let u and U^n be solutions of (1) and (6) respectively, with u_{0h} chosen such that $\|u_0 - u_{0h}\| \leq ch^2$. Under the required regularity (H1)-(H3), there exists a constant c such that, for $t_n \in J$ and small τ , we have*

$$\|U^n - u(t_n)\| \leq c(u)(h^2 + \tau).$$

Moreover, for sufficiently small τ , h and $\frac{\tau}{h}$, there is a unique solution U^n of the complete discrete scheme (6).

PROOF. We use the partitioning of the error

$$U^n - u^n = (U^n - \tilde{U}^n) + (\tilde{U}^n - u^n) = \theta_n + \rho_n, \tag{9}$$

with $u^n = u(t_n)$, $\tilde{U}^n = \tilde{u}_h(t_n)$, where \tilde{u}_h is the elliptic projection of u^n defined by (5). By virtue of Lemma 1, it suffices to bound θ_n . We have for $\chi \in S_h$ that

$$\begin{aligned}
&(\partial_n \theta_n, \chi) + (k(U^n) \nabla \theta_n, \nabla \chi) \\
&= (\partial_n U^n, \chi) + (k(U^n) \nabla U^n, \nabla \chi) - (\partial_n \tilde{U}^n, \chi) - (k(U^n) \nabla \tilde{U}^n, \nabla \chi) \\
&= \frac{\lambda}{\left(\int_{\Omega} f(U^n) dx\right)^2} (f(U^n), \chi) - (u_t^n, \chi) - (\partial_n \tilde{U}^n - u_t^n, \chi) \\
&\quad - (k(u^n) \nabla \tilde{U}^n, \nabla \chi) - ((k(U^n) - k(u^n)) \nabla \tilde{U}^n, \nabla \chi)
\end{aligned}$$

and, in view of the elliptic projection (5) and the equation of the continuous problem (1), we can write:

$$\begin{aligned}
& (\partial_n \theta_n, \chi) + (k(U^n) \nabla \theta_n, \nabla \chi) \\
&= \frac{\lambda}{\left(\int_{\Omega} f(U^n) dx \right)^2} (f(U^n), \chi) - \frac{\lambda}{\left(\int_{\Omega} f(u^n) dx \right)^2} (f(u^n), \chi) \\
&\quad - (\partial_n \rho_n, \chi) - (\partial_n u^n - u_t^n, \chi) - ((k(U^n) - k(u^n)) \nabla \tilde{U}^n, \nabla \chi) \\
&= \frac{\lambda}{\left(\int_{\Omega} f(U^n) dx \right)^2} (f(U^n) - f(u^n), \chi) \\
&\quad + \left(\frac{\lambda}{\left(\int_{\Omega} f(U^n) dx \right)^2} - \frac{\lambda}{\left(\int_{\Omega} f(u^n) dx \right)^2} \right) (f(u^n), \chi) \\
&\quad - (\partial_n \rho_n, \chi) - (\partial_n u^n - u_t^n, \chi) - ((k(U^n) - k(u^n)) \nabla \tilde{U}^n, \nabla \chi).
\end{aligned}$$

Choosing $\chi = \theta_n$ and using the fact that $\nabla \tilde{U}^n$, u^n , U^n are bounded, we get:

$$\begin{aligned}
& \frac{1}{2} \partial_n \|\theta_n\|^2 + k_1 \|\nabla \theta_n\|^2 \\
& \leq c \|U^n - u^n\| (\|\theta_n\| + \|\nabla \theta_n\|) + (\|\partial_n \rho_n\| + \|\partial_n u^n - u_t^n\|) \|\theta_n\|.
\end{aligned}$$

Hence, by Young's inequality,

$$\partial_n \|\theta_n\|^2 + k_1 \|\nabla \theta_n\|^2 \leq c (\|\theta_n\|^2 + \|\rho_n\|^2 + \|\partial_n \rho_n\|^2 + \|\partial_n u^n - u_t^n\|^2). \quad (10)$$

Introducing the notation $R_n = \|\rho_n\|^2 + \|\partial_n \rho_n\|^2 + \|\partial_n u^n - u_t^n\|^2$ we write (10) in the form $\partial_n \|\theta_n\|^2 + k_1 \|\nabla \theta_n\|^2 \leq c (\|\theta_n\|^2 + R_n)$, and it results that

$$(1 - c\tau) \|\theta_n\|^2 \leq \|\theta_{n-1}\|^2 + c\tau R_n.$$

For $\tau < \frac{1}{c}$, we have $\|\theta_n\|^2 \leq \frac{1}{1-c\tau} \|\theta_{n-1}\|^2 + \frac{c\tau R_n}{1-c\tau}$. Using the fact that $\frac{1}{1-c\tau} \approx 1 + c\tau$ for τ sufficiently small, it follows that $\|\theta_n\|^2 \leq (1 + c\tau) \|\theta_{n-1}\|^2 + c\tau R_n$. By induction, we get:

$$\begin{aligned}
\|\theta_n\|^2 & \leq (1 + c\tau)^n \|\theta_0\|^2 + c\tau \sum_{j=1}^n (1 + c\tau)^{n-j} R_j \\
& \leq c \|\theta_0\|^2 + c\tau \sum_{j=1}^n R_j, \text{ for } t_n \in J.
\end{aligned} \quad (11)$$

We now recall that from Lemma 1

$$\|\rho_j\| \leq c(u) h^2, \quad \|\partial_n \rho_j\| = \left\| \frac{1}{\tau} \int_{t_{j-1}}^{t_j} \rho_t ds \right\| \leq c(u) h^2.$$

On the other hand, we have

$$\|\partial_n u^j - u_t^j\|^2 = \left\| \frac{1}{\tau} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s) ds \right\|^2 \leq c(u) \tau.$$

It yields that $R_j \leq c(u)(h^2 + \tau)^2$. Taking the above estimates together with (11) and (7) we prove the intended bound for θ_n :

$$\|\theta_n\| \leq c\|u_{0h} - u_0\| + c(u)(h^2 + \tau) \leq c(u)(h^2 + \tau).$$

It remains to prove the second part of the theorem (uniqueness). Let $U^n = X$ and $U^n = Y$ be two solutions of the fully discrete problem:

$$\begin{aligned} & (X - Y, \chi) + \tau(k(X)\nabla X - k(Y)\nabla Y, \nabla \chi) \\ &= \frac{\lambda\tau}{(f f(X))^2}(f(X) - f(Y), \chi) + \left(\frac{\lambda\tau}{(f f(X))^2} - \frac{\lambda\tau}{(f f(Y))^2} \right) (f(Y), \chi). \end{aligned}$$

Taking $\chi = X - Y$, we have

$$\begin{aligned} & \|X - Y\|^2 + \tau(k(X)\nabla(X - Y), \nabla(X - Y)) = \frac{\lambda\tau}{(f f(X))^2}(f(X) - f(Y), X - Y) \\ &+ \left(\frac{\lambda\tau}{(f f(X))^2} - \frac{\lambda\tau}{(f f(Y))^2} \right) (f(Y), X - Y) - \tau((k(X) - k(Y))\nabla Y, \nabla(X - Y)). \end{aligned}$$

Thus,

$$\|X - Y\|^2 + \frac{1}{2}k_1\tau\|\nabla(X - Y)\|^2 \leq c\|X - Y\|^2(\tau + \tau\|\nabla Y\|_{L^\infty}^2).$$

According to Lemma 2 we have

$$\|\nabla Y\|_{L^\infty} \leq \|\nabla \tilde{u}_h\|_{L^\infty} + \|\nabla \theta_n\|_{L^\infty} \leq c + ch^{-1}\|\nabla \theta_n\|.$$

Taking into account the estimate for R_n , we get

$$k_1\|\nabla \theta_n\|^2 \leq c\|\theta_{n-1}\|^2 + \tau\|\theta_n\|^2 + \tau R_n \leq c(h^2 + \tau)^2,$$

and we deduce that

$$\tau\|\nabla Y\|_{L^\infty}^2 \leq c\left(\tau + h^2 + \left(\frac{\tau}{h}\right)^2\right).$$

Then, for sufficient small τ , h and $\frac{\tau}{h}$, we get the uniqueness of the solution of the complete discrete scheme.

5 The Crank-Nicolson-Galerkin scheme

This section is devoted to the study of the following Crank-Nicolson-Galerkin scheme:

$$\begin{aligned} (\partial_n U^n, \chi) + (k(\bar{U}^n) \nabla \bar{U}^n, \nabla \chi) &= \frac{\lambda}{\left(\int_{\Omega} f(\bar{U}^n) dx \right)^2} (f(\bar{U}^n), \chi), \quad \forall \chi \in S_h, t_n \in J, \\ U^0 &= u_{0h}, \end{aligned} \tag{12}$$

with $\bar{U}^n = \frac{U^n + U^{n-1}}{2}$. Before proceeding to the main result of the section – an $o(h^2 + \tau^2)$ error bound – we need an auxiliary estimate.

Lemma 6 *Besides the hypotheses (H1)-(H3), let us further assume*

$$(\mathbf{H4}) \quad u_{tt} \in L^\infty(0, T, H^1(\Omega)).$$

Then, $\|\nabla \tilde{u}_{h,tt}\| \leq c(u)$, where \tilde{u}_h is the elliptic projection defined by (5).

PROOF. Differentiating (5) twice in time we find that

$$(k(u) \nabla \tilde{u}_{h,tt}, \nabla \chi) = (k(u) \nabla u_{tt}, \nabla \chi) - 2(k'(u) \nabla \rho_t, \nabla \chi) - (k''(u) \nabla \rho, \nabla \chi).$$

Taking $\chi = \tilde{u}_{h,tt}$, it follows that

$$k_1 \|\nabla \tilde{u}_{h,tt}\|^2 \leq c(u) (\|\nabla u_{tt}\| + \|\nabla \rho_t\| + \|\nabla \rho\|) \|\nabla \tilde{u}_{h,tt}\|$$

and, recalling Lemma 1 and (H4), we obtain the intended conclusion.

Theorem 7 *Let u and U^n be respectively the solutions of (1) and (12). Then, under the hypotheses (H1)-(H4), we have $\|U^n - u(t_n)\| \leq c(u)(h^2 + \tau^2)$, $t_n \in J$.*

PROOF. Splitting the error by means of the elliptic projection as in (9), one gets

$$\begin{aligned}
& (\partial_n \theta^n, \chi) + (k(\bar{U}^n) \nabla \bar{\theta}^n, \nabla \chi) \\
&= (\partial_n U^n, \chi) + (k(\bar{U}^n) \nabla \bar{U}^n, \nabla \chi) - (\partial_n \tilde{U}^n, \chi) - (k(\bar{U}^n) \nabla \bar{\tilde{U}}^n, \nabla \chi) \\
&= \frac{\lambda}{\left(\int_{\Omega} f(\bar{U}^n) dx\right)^2} (f(\bar{U}^n), \chi) - \left(u_t^{n-\frac{1}{2}}, \chi\right) - \left(\partial_n \tilde{U}^n - u_t^{n-\frac{1}{2}}, \chi\right) \\
&\quad - \left(k(u^{n-\frac{1}{2}}) \nabla \tilde{U}^{n-\frac{1}{2}}, \nabla \chi\right) - \left(k(\bar{U}^n) \nabla \bar{\tilde{U}}^n - k(u^{n-\frac{1}{2}}) \nabla \tilde{U}^{n-\frac{1}{2}}, \nabla \chi\right) \\
&= \frac{\lambda}{\left(\int_{\Omega} f(\bar{U}^n) dx\right)^2} (f(\bar{U}^n), \chi) - \frac{\lambda}{\left(\int_{\Omega} f(u^{n-\frac{1}{2}}) dx\right)^2} (f(u^{n-\frac{1}{2}}), \chi) \\
&\quad - \left(\partial_n \tilde{U}^n - u_t^{n-\frac{1}{2}}, \chi\right) - \left((k(\bar{U}^n) - k(u^{n-\frac{1}{2}})) \nabla \bar{\tilde{U}}^n\right. \\
&\quad \left.+ k(u^{n-\frac{1}{2}}) \nabla (\bar{\tilde{U}}^n - \tilde{U}^{n-\frac{1}{2}}), \nabla \chi\right).
\end{aligned}$$

Setting $\chi = \bar{\theta}^n$, it follows from $(\partial_n \theta^n, \bar{\theta}^n) = \frac{1}{2} \bar{\partial} \|\theta^n\|^2$ and (H1)-(H3) that

$$\begin{aligned}
\frac{1}{2} \bar{\partial} \|\theta^n\|^2 + \mu \|\nabla \bar{\theta}^n\|^2 &\leq \frac{\lambda}{\left(\int_{\Omega} f(u^{n-\frac{1}{2}}) dx\right)^2} |(f(u^{n-\frac{1}{2}}) - f(\bar{U}^n), \nabla \bar{\theta}^n)| \\
&\quad + \left| \frac{\lambda}{\left(\int_{\Omega} f(u^{n-\frac{1}{2}}) dx\right)^2} - \frac{\lambda}{\left(\int_{\Omega} f(\bar{U}^n) dx\right)^2} (f(\bar{U}^n), \nabla \bar{\theta}^n) \right| \\
&\quad + c \left(\|\partial_n \tilde{U}^n - u_t^{n-\frac{1}{2}}\| + \|\bar{U}^n - u^{n-\frac{1}{2}}\| + \|\nabla(\bar{\tilde{U}}^n - \tilde{U}^{n-\frac{1}{2}})\| \right) \|\nabla \bar{\theta}^n\| \\
&\leq c \left(\|\partial_n \tilde{U}^n - u_t^{n-\frac{1}{2}}\| + \|\bar{U}^n - u^{n-\frac{1}{2}}\| + \|\nabla(\bar{\tilde{U}}^n - \tilde{U}^{n-\frac{1}{2}})\| \right) \|\nabla \bar{\theta}^n\|.
\end{aligned}$$

Young's inequality gives

$$\bar{\partial} \|\theta^n\|^2 \leq c \left(\|\partial_n \tilde{U}^n - u_t^{n-\frac{1}{2}}\|^2 + \|\bar{U}^n - u^{n-\frac{1}{2}}\|^2 + \|\nabla(\bar{\tilde{U}}^n - \tilde{U}^{n-\frac{1}{2}})\|^2 \right). \quad (13)$$

Estimating each term on the right hand side of the inequality (13) separately, we have

$$\|\bar{U}^n - u^{n-\frac{1}{2}}\| \leq \|\bar{\theta}^n\| + \|\bar{\rho}^n\| + \|\bar{u}^n - u^{n-\frac{1}{2}}\| \leq \|\bar{\theta}^n\| + c(u)(h^2 + \tau^2), \quad (14)$$

$$\|\partial_n \tilde{U}^n - u_t^{n-\frac{1}{2}}\| \leq \|\partial_n \rho^n\| + \|\partial_n u^n - u_t^{n-\frac{1}{2}}\| \leq c(u)(h^2 + \tau^2), \quad (15)$$

and by Lemma 6

$$\left\| \nabla(\bar{\tilde{U}}^n - \tilde{U}^{n-\frac{1}{2}}) \right\| \leq c\tau \int_{t_{n-1}}^{t_n} \|\nabla \tilde{u}_{h,tt}\| ds \leq c(u)\tau^2. \quad (16)$$

Inequalities (13)–(16) together show that

$$\begin{aligned}\bar{\partial}\|\theta^n\|^2 &\leq c\|\bar{\theta}^n\|^2 + c(u)(h^2 + \tau^2)^2 \leq \frac{c}{4}\|\theta^n + \theta^{n-1}\|^2 + c(u)(h^2 + \tau^2)^2 \\ &\leq c\|\theta^n\|^2 + c\|\theta^{n-1}\|^2 + c(u)(h^2 + \tau^2)^2,\end{aligned}$$

which gives $(1 - c\tau)\|\theta^n\|^2 \leq (1 + c\tau)\|\theta^{n-1}\|^2 + c(u)\tau(h^2 + \tau^2)^2$. Then, by induction, we have that for small τ

$$\|\theta^n\| \leq c\|\theta^0\| + c(u)(h^2 + \tau^2) \leq c\|u_{0h} - u_0\| + c(u)(h^2 + \tau^2) \text{ for } t_n \in J.$$

Or $\|u_{0h} - u_0\| \leq ch^2$, and then $\|\theta^n\| \leq c(u)(h^2 + \tau^2)$, which leads, in view of Lemma 1, to the intended result.

6 Method of solution

We divide the interval $\Omega = (-1, 1)$ into N equal finite elements: $x_0 = -1 < x_1 < \dots < x_N = 1$. Let (x_j, x_{j+1}) be a partition of Ω and $x_{j+1} - x_j = h = \frac{1}{N}$ the step length. By S we denote a basis of the usual pyramid functions:

$$v_j(x) = \begin{cases} \frac{1}{h}x + (1 - j) & \text{on } [x_{j-1}, x_j], \\ -\frac{1}{h}x + (1 + j) & \text{on } [x_j, x_{j+1}], \\ 0 & \text{otherwise.} \end{cases}$$

We first write (1) in variational form with $k = 1$. We then multiply (1) by v_j (for j fixed). We have, using the boundary conditions, that

$$\int_{\Omega} \frac{\partial u}{\partial t} v_j dx + \int_{\Omega} \nabla u \nabla v_j dx = \lambda \frac{\int_{\Omega} f(u) v_j dx}{\left(\int_{\Omega} f(u) dx\right)^2}.$$

By the Crank-Nicolson approach we obtain

$$\int_{\Omega} u^{n+1} v_j dx + \tau \int_{\Omega} \nabla u^{n+1} \nabla v_j dx = \int_{\Omega} u^n v_j + \lambda \tau \frac{\int_{\Omega} f(u^n) v_j dx}{\left(\int_{\Omega} f(u^n) dx\right)^2}, \quad (17)$$

so the approximation u^{n+1} to the function u can be written as

$$u^{n+1} = \sum_{i=-1}^N \alpha_i^{n+1} v_i,$$

where the α_i^{n+1} are unknown real coefficients to be determined. From (17) it is easy to obtain the following system of $(N - 1)$ linear algebraic equations:

$$\begin{aligned} \left(\frac{h}{6} - \frac{\tau}{h}\right) \alpha_{j-1}^{n+1} + \left(\frac{2h}{3} + \frac{\tau}{2h}\right) \alpha_j^{n+1} + \left(\frac{h}{6} - \frac{\tau}{h}\right) \alpha_{j+1}^{n+1} \\ = \frac{h}{6} \alpha_{j-1}^n + \frac{2h}{3} \alpha_j^n + \frac{h}{6} \alpha_{j+1}^n + \frac{\lambda \tau \int_{\Omega} f(u^n) v_j dx}{\left(\int_{\Omega} f(u^n) dx\right)^2}. \end{aligned} \quad (18)$$

Using the Dirichlet boundary conditions, we have

$$\begin{aligned} \alpha_{-1}^{n+1} &= \alpha_1^{n+1} - (1 + h) \alpha_0^{n+1}, \\ \alpha_{-1}^n &= \alpha_1^n - (1 + h) \alpha_0^n, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \alpha_N^{n+1} &= c \alpha_{N-1}^{n+1}, \\ \alpha_N^n &= c \alpha_{N-1}^n, \end{aligned} \quad (20)$$

where $c = \frac{Nh}{(N-1)h-1}$. Substituting the expressions (19) and (20) into (18), we obtain the system of equations

$$\begin{aligned} (b - (1 + h)) \alpha_0^{n+1} + 2a \alpha_1^{n+1} \\ = \left(\frac{2h}{3} - (1 + h)\right) \alpha_0^n + \frac{h}{3} \alpha_1^n + \frac{\lambda \tau \int_{\Omega} f(u^n) v_0 dx}{\left(\int_{\Omega} f(u^n) dx\right)^2}, \quad j = 0 \\ a \alpha_{j-1}^{n+1} + b \alpha_j^{n+1} + a \alpha_{j+1}^{n+1} \\ = \frac{h}{6} \alpha_{j-1}^n + \frac{2h}{3} \alpha_j^n + \frac{h}{6} \alpha_{j+1}^n + \frac{\lambda \tau \int_{\Omega} f(u^n) v_j dx}{\left(\int_{\Omega} f(u^n) dx\right)^2}, \quad j = 1 \dots N - 2, \\ a \alpha_{N-2}^{n+1} + (ac + b) \alpha_{N-1}^{n+1} \\ = \frac{h}{6} \alpha_{N-2}^n + \frac{h}{6} (c + 4) \alpha_{N-1}^n + \frac{\lambda \tau \int_{\Omega} f(u^n) v_{N-1} dx}{\left(\int_{\Omega} f(u^n) dx\right)^2}, \quad j = N - 1, \end{aligned} \quad (21)$$

where

$$a = \frac{h}{6} - \frac{\tau}{h}, \quad b = \frac{2h}{3} + \frac{\tau}{2h},$$

and thus a simple algorithm for solving the fully discrete problem.

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